

A Simple Construction of Recursion Operators for Multidimensional Dispersionless Integrable Systems

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We present a simple novel construction of recursion operators for integrable multidimensional dispersionless systems that admit Lax-type representations whose operators are linear in the spectral parameter and do not involve the derivatives with respect to the latter. New examples of recursion operators obtained using our technique include *inter alia* those for the general heavenly equation, which describes a class of anti-self-dual solutions of the vacuum Einstein equations, and a six-dimensional equation resulting from a system of Ferapontov and Khusnutdinova.

Introduction

The field of integrable systems is a dynamic subject closely related to many parts of modern mathematics and featuring manifold applications, cf. e.g. [1, 3, 5, 7, 9, 12, 20, 36, 37, 40, 47] and references therein.

A characteristic feature of integrable systems is that they never show up alone – instead, they always belong to integrable hierarchies, and the latter are typically constructed using the recursion operators (ROs), cf. e.g. [1, 3, 12, 21, 36, 39, 40] and references therein. In particular, the ROs for multidimensional integrable dispersionless systems are now a subject of intense research, cf. e.g. [1, 12, 24, 28, 30, 33, 34, 35, 40] and references therein. In fact, the overwhelming majority of integrable systems in four or more independent variable known to date, including those relevant for applications, like the (anti-)self-dual Yang–Mills equations or the (anti-)self-dual vacuum Einstein equations, is dispersionless, i.e., can be rewritten as first-order homogeneous quasilinear systems, cf. e.g. [1, 12, 45] and Example 5 below for details.

The study of a hierarchy associated to a given integrable system, and hence of its recursion operator, is important for many reasons, *inter alia* because known exact solutions of integrable systems, like, say, multisoliton and finite-gap solutions, always happen to be invariant under a certain combination of the flows in the hierarchy, and this is instrumental in the actual construction of the solutions in questions, cf. e.g. [12, 36].

Another feature making recursion operators relevant for applications is that existence of a (formal) recursion operator can in many situations be employed as an efficient integrability test or as a means of proving nonintegrability, see e.g. [32] and references therein for general theory and [46] for a recent application.

There are several methods for construction of the ROs for the integrable dispersionless systems: the partner symmetry approach [24], the method based on adjoint action of the Lax operators [30], and the approach using the Cartan equivalence method [33, 34, 35].

Below we present another method for construction of ROs for integrable multidimensional dispersionless systems. Roughly speaking, it consists in constructing a Lax-type representation for the system under study with the following additional properties: i) the Lax operators are linear in the spectral parameter and ii) the solution of the associated linear system is a (nonlocal) symmetry of the nonlinear

system under study. Taking the coefficients at the powers of spectral parameter in the operators constituting the newly constructed Lax-type representation yields, under certain technical conditions, the RO. A similar approach, with symmetries replaced by cosymmetries and Proposition 1 by Proposition 2, allows one to look for the adjoint ROs.

To the best of author's knowledge, the key idea of our approach, i.e., that the Lax-type representation employed for the construction of the recursion operator should be *constructed* from the original Lax-type representation but is by no means obliged to be identical with the latter, has not yet appeared in the literature. This kind of flexibility noticeably enhances the applicability of our method. On the other hand, the extraction of the RO from a Lax-type representation which is linear in the spectral parameter was already explored to some extent e.g. in [41].

The paper is organized as follows. In Section 1 we recall some basic facts concerning the geometric theory of PDEs. In Section 2 we state some general results on the construction of recursion operators. Section 3, which is the core of the paper, explores the construction of special Lax-type representations that give rise to ROs through the results of Section 2. Section 4 gives a selection of examples illustrating our approach, and Section 5 provides a brief discussion.

1 Preliminaries

Let $\mathcal{F} = 0$ be a system of m PDEs $F_I = 0$, $I = 1, \dots, m$, in d independent variables x^i , $i = 1, \dots, d$, so we put $\vec{x} = (x^1, \dots, x^d)$, for an unknown N -component vector function $\mathbf{u} = (u^1, \dots, u^N)^T$, where the superscript ' T ' indicates the transposed matrix. Let

$$u_{i_1 \dots i_n}^\alpha = \partial^{i_1 + \dots + i_n} u^\alpha / \partial (x^1)^{i_1} \dots \partial (x^n)^{i_n}$$

and $u_{0 \dots 0}^\alpha \equiv u^\alpha$. In what follows all functions are tacitly assumed to be meromorphic in all their arguments.

As e.g. in [40, 17], x^i and $u_{i_1 \dots i_n}^\alpha$ are considered here and below as independent quantities and can be viewed as coordinates on an abstract infinite-dimensional space (a jet space). By a *local function* we shall mean a function of \vec{x} , \mathbf{u} and of a finite number of the derivatives of \mathbf{u} . To indicate locality of a function f we shall write it as $f(\vec{x}, [\mathbf{u}])$.

We denote by

$$D_{x^j} = \frac{\partial}{\partial x^j} + \sum_{\alpha=1}^N \sum_{i_1, \dots, i_n=0}^{\infty} u_{i_1 \dots i_{j-1}(i_j+1)i_{j+1} \dots i_n}^\alpha \frac{\partial}{\partial u_{i_1 \dots i_n}^\alpha}$$

the total derivatives with respect to x^j , cf. e.g. [40, 17]. For the sake of simplicity the extensions of D_{x^i} to nonlocal variables are again denoted by D_{x^i} , as e.g. in [30].

The system $\mathcal{F} = 0$ together with all its differential consequences

$$D_{x^1}^{i_1} \dots D_{x^n}^{i_n} \mathcal{F} = 0, \quad i_1, i_2, \dots, i_n = 1, \dots, d, \quad n = 1, 2, \dots,$$

defines a *diffiety* $\text{Sol}_{\mathcal{F}}$, a submanifold of the infinite jet space, see e.g. [17] and references therein for details. This submanifold can be informally thought of as a set of all formal solutions of $\mathcal{F} = 0$, which motivates the choice of notation. Below all equations will be required to hold only on $\text{Sol}_{\mathcal{F}}$, or a certain differential covering thereof, see e.g. [17] for details on the latter, rather than e.g. on the whole jet space unless otherwise explicitly stated.

The *directional derivative* (cf. e.g. [3, 30] and references therein) along $\mathbf{U} = (U^1, \dots, U^N)^T$ is the vector field on the jet space

$$\partial_{\mathbf{U}} = \sum_{\alpha=1}^N \sum_{i_1, \dots, i_n=0}^{\infty} (D_{x^1}^{i_1} \dots D_{x^n}^{i_n} U^\alpha) \frac{\partial}{\partial u_{i_1 \dots i_n}^\alpha}.$$

The total derivatives as well as the directional derivative can be applied to (possibly vector or matrix) local functions P .

Recall, cf. e.g. [40, 17], that a local N -component vector function \mathbf{U} is a (characteristic of a) *symmetry* for the system $\mathcal{F} = 0$ if \mathbf{U} satisfies the linearized version of this system, namely, $\ell_{\mathcal{F}}(\mathbf{U}) = 0$. We stress that by our blanket assumption this is required to hold on $\text{Sol}_{\mathcal{F}}$ only rather than on the whole jet space. Informally this definition means that the flow $\mathbf{u}_{\tau} = \mathbf{U}$ leaves $\text{Sol}_{\mathcal{F}}$ invariant.

Here

$$\ell_f = \sum_{\alpha=1}^N \sum_{i_1, \dots, i_n=0}^{\infty} \frac{\partial f}{\partial u_{i_1 \dots i_n}^{\alpha}} D_{x^1}^{i_1} \dots D_{x^n}^{i_n}$$

is the operator of linearization and $\ell_{\mathcal{F}} = (\ell_{F_1}, \dots, \ell_{F_m})^T$. Notice an important identity $\ell_f(\mathbf{U}) = \partial_{\mathbf{U}}(f)$ which holds for any local f and \mathbf{U} , see e.g. [17].

Finally, recall that if we have an operator in total derivatives

$$Q = Q^0 + \sum_{k=1}^q \sum_{j_1=1}^d \dots \sum_{j_k=1}^d Q^{j_1 \dots j_k} D_{x^{j_1}} \dots D_{x^{j_k}},$$

where Q^0 and $Q^{j_1 \dots j_k}$ are $s \times s'$ -matrix-valued local functions, its formal adjoint reads (cf. e.g. [40])

$$Q^{\dagger} = (Q^0)^T + \sum_{k=1}^q \sum_{j_1=1}^d \dots \sum_{j_k=1}^d (-1)^k D_{x^{j_1}} \dots D_{x^{j_k}} \circ (Q^{j_1 \dots j_k})^T.$$

For integrable systems in more than two independent variables their symmetries usually depend, in addition to local variables, described above, also on nonlocal variables. In other words, we are forced to consider solutions \mathbf{U} of $\ell_{\mathcal{F}}(\mathbf{U}) = 0$ involving nonlocal variables. A number of authors refers to such objects as to the shadows of nonlocal symmetries, see e.g. [17] and references therein for more details on this terminology; cf. also [5] and references therein. However, as in the present paper we shall not deal with full nonlocal symmetries in the sense of [17], for the sake of simplicity in what follows we shall refer to solutions \mathbf{U} of $\ell_{\mathcal{F}}(\mathbf{U}) = 0$ involving nonlocal variables just as to *nonlocal symmetries*.

A *cosymmetry*, also known as adjoint symmetry, see e.g. [5], γ is a quantity which is dual to a symmetry: it has m rather than N components and satisfies (cf. e.g. [3, 17] for details) the system

$$\ell_{\mathcal{F}}^{\dagger}(\gamma) = 0.$$

Note that cosymmetries, just like symmetries, may depend on nonlocal variables in addition to the local ones.

A *recursion operator* (RO) for our system $\mathcal{F} = 0$ is then, roughly speaking, an operator that sends any symmetry of $\mathcal{F} = 0$ into another symmetry thereof [39, 40]. However, this definition works well only when the RO contains no nonlocal terms, as often is the case for linear and linearizable equations, cf. e.g. [4, 40] and references therein.

In order to properly handle nonlocal terms in ROs it is more appropriate to view the RO as a Bäcklund auto-transformation for $\ell_{\mathcal{F}}(\mathbf{G}) = 0$, see [30, 41, 42] and references therein for details, and below we adhere to this point of view. Likewise, an *adjoint recursion operator* is from this perspective a Bäcklund auto-transformation for $\ell_{\mathcal{F}}^{\dagger}(\gamma) = 0$, cf. e.g. [17].

As a closing remark, recall that a partial differential system is called *dispersionless* (cf. e.g. [28] and references therein) if it can be written in the form of a quasilinear homogeneous first-order system

$$\sum_{j=1}^d \sum_{\alpha=1}^n A_{I\alpha}^j(\mathbf{u}) \frac{\partial u^{\alpha}}{\partial x^j} = 0, \quad (1)$$

where $I = 1, \dots, m$, $m \geq N$, and $\mathbf{u} = (u^1, \dots, u^N)^T$.

This class of systems is quite rich: for instance, quasilinear scalar second-order PDEs which do not explicitly involve the dependent variable u ,

$$\sum_{i=1}^d \sum_{j=i}^d f^{ij}(\vec{x}, \partial u / \partial x^1, \dots, \partial u / \partial x^d) \partial^2 u / \partial x^i \partial x^j = 0,$$

can be brought into the form (1) by putting $\mathbf{u} = (\partial u / \partial x^1, \dots, \partial u / \partial x^d)^T$.

2 Some remarks on recursion operators

Consider the following differential operators in total derivatives

$$\begin{aligned} A_i &= A_i^0 + \sum_{j=1}^d A_i^j D_{x^j}, & B_i &= B_i^0 + \sum_{j=1}^d B_i^j D_{x^j}, & i &= 1, 2, \\ L &= L^0 + \sum_{k=1}^d L^k D_{x^k}, & M &= M^0 + \sum_{k=1}^d M^k D_{x^k}, \end{aligned} \quad (2)$$

where $A_i^j = A_i^j(\vec{x}, [\mathbf{u}])$ and $B_i^j = B_i^j(\vec{x}, [\mathbf{u}])$ for $i = 1, 2$ and $j = 1, \dots, d$ are scalar functions, $A_i^0 = A_i^0(\vec{x}, [\mathbf{u}])$ and $B_i^0 = B_i^0(\vec{x}, [\mathbf{u}])$ are $N \times N$ matrices, $L^0 = L^0(\vec{x}, [\mathbf{u}])$ and $L^k = L^k(\vec{x}, [\mathbf{u}])$ for $k = 1, \dots, d$ are $N \times m$ matrices, while $M^0 = M^0(\vec{x}, [\mathbf{u}])$ and $M^k = M^k(\vec{x}, [\mathbf{u}])$ for $k = 1, \dots, d$ are $m \times N$ matrices.

Proposition 1. *Let the operators A_i, B_i, L, M of the form (2) be such that*

$$i) [A_1, A_2] = 0, \quad (3)$$

$$ii) (A_1 B_2 - A_2 B_1) = L \circ \ell_{\mathcal{F}}, \quad (4)$$

$$iii) \ell_{\mathcal{F}} = M \circ (B_1 A_2 - B_2 A_1), \quad (5)$$

iv) *there exist two distinct indices $p, q \in \{1, \dots, d\}$ such that we can express $D_{x^p} \tilde{\mathbf{U}}$ and $D_{x^q} \tilde{\mathbf{U}}$ from the relations*

$$A_i(\tilde{\mathbf{U}}) = B_i(\mathbf{U}), \quad i = 1, 2. \quad (6)$$

Then relations (6) define a recursion operator for $\mathcal{F} = 0$, i.e., whenever $\mathbf{U} = (U^1, \dots, U^N)^T$ is a (possibly nonlocal) symmetry for $\mathcal{F} = 0$, so is $\tilde{\mathbf{U}} = (\tilde{U}^1, \dots, \tilde{U}^N)^T$ defined by (6).

Proof. First of all, if $\mathbf{U} = (U^1, \dots, U^N)^T$ is a symmetry for $\mathcal{F} = 0$, then the system (6) for $\tilde{\mathbf{U}}$ is compatible by virtue of i)–iii).

We now have $\ell_{\mathcal{F}}(\tilde{\mathbf{U}}) = 0$ by virtue of (5), and hence $\tilde{\mathbf{U}}$ is a shadow of symmetry for our system. \square

As an aside note that the condition (5) in a slightly different form has appeared in [11, 15, 16], and was given there a certain geometric interpretation.

In complete analogy with the above we can also readily prove the counterpart of Proposition 1 for adjoint recursion operators.

Proposition 2. *Suppose that A_i, B_i, L, M are as before, but with A_i^0 and B_i^0 being $m \times m$ matrices. Further assume that these operators are such that*

$$i) [A_1, A_2] = 0, \quad (7)$$

$$ii) \ell_{\mathcal{F}}^\dagger = L \circ (B_1 A_2 - B_2 A_1), \quad (8)$$

$$iii) (A_1 B_2 - A_2 B_1) = M \circ \ell_{\mathcal{F}}^\dagger, \quad (9)$$

iv) *there exist two distinct indices $p, q \in \{1, \dots, d\}$ such that we can express $D_{x^p} \tilde{\gamma}$ and $D_{x^q} \tilde{\gamma}$ from the system*

$$A_i(\tilde{\gamma}) = B_i(\gamma), \quad i = 1, 2. \quad (10)$$

Then (10) defines an adjoint recursion operator for $\mathcal{F} = 0$, i.e., whenever $\gamma = (\gamma^1, \dots, \gamma^m)^T$ is a cosymmetry for $\mathcal{F} = 0$, then so is $\tilde{\gamma}$ defined by (10).

3 Recursion operators from Lax-type representations

We start with the following observation which is readily checked by straightforward computation.

Proposition 3. *Suppose that the operators A_i and B_i define a RO (resp. an adjoint RO) as in Proposition 1 (resp. as in Proposition 2). Then the operators $\mathcal{L}_i = \lambda A_i - B_i$, $i = 1, 2$, where λ is a spectral parameter, satisfy $[\mathcal{L}_1, \mathcal{L}_2] = 0$, i.e., they constitute a Lax-type representation for $\mathcal{F} = 0$.*

Thus, if our system admits an (adjoint) RO, it also admits a Lax-type representation which is linear in the spectral parameter.

Hence, a natural source of A_i, B_i and L, M satisfying the conditions of Proposition 1 is provided by the Lax-type representations for $\mathcal{F} = 0$ of the form

$$\mathcal{L}_i \psi = 0, \quad i = 1, 2, \quad (11)$$

with \mathcal{L}_i linear in λ such that ψ is a nonlocal symmetry of $\mathcal{F} = 0$, i.e., we have $\ell_{\mathcal{F}}(\psi) = 0$, cf. e.g. [41]. Then putting $\mathcal{L}_i = \lambda B_i - A_i$ or $\mathcal{L}_i = \lambda A_i - B_i$ gives us natural candidates for A_i and B_i which then should be checked against the conditions of Proposition 1, and, if the latter hold, yield a recursion operator for $\mathcal{F} = 0$.

Even if it is impossible to construct the recursion operator from (11), one still can construct infinite series of nonlocal symmetries for $\mathcal{F} = 0$ expanding ψ into formal Taylor or Laurent series in λ , cf. e.g. [41, 44].

It is important to stress that the Lax representation with the operators \mathcal{L}_i employed in the above construction does not have to be the *original* Lax representation of our system. In general, we should *custom tailor* the operators $\mathcal{L}_{1,2}$ constituting the Lax pair for the construction in question, so that the solutions of the associated linear problem (11) are symmetries, i.e., satisfy the linearized version of our system.

The natural building blocks for these \mathcal{L}_i are the original Lax operators \mathcal{X}_i , their formal adjoints \mathcal{X}_i^\dagger , and the operators $\text{ad}_{\mathcal{X}_i} = [\mathcal{X}_i, \cdot]$, but in general one has to twist them, cf. e.g. Example 4 below.

More explicitly, suppose that the system under study, i.e., $\mathcal{F} = 0$, admits a Lax representation of the form

$$\mathcal{X}_i \psi = 0, \quad i = 1, 2, \quad (12)$$

where \mathcal{X}_i are linear in the spectral parameter λ but such ψ does *not* satisfy $\ell_{\mathcal{F}}(\psi) = 0$, i.e., ψ is not a (nonlocal) symmetry.

Then we can seek for a nonlocal symmetry of $\mathcal{F} = 0$ of the form

$$\Phi = \Phi(\vec{x}, [\mathbf{u}, \psi, \chi, \vec{\zeta}]), \quad (13)$$

i.e., for a vector function of \vec{x} , and of \mathbf{u} , ψ , χ , and $\vec{\zeta}$ and a finite number of the derivatives of \mathbf{u} , ψ , χ , and $\vec{\zeta}$.

Here χ satisfies the system

$$\mathcal{X}_i^\dagger \chi = 0, \quad i = 1, 2,$$

$\mathcal{Z} = \sum_{j=1}^d \zeta^j \partial / \partial x^j$ satisfies

$$[\mathcal{X}_i, \mathcal{Z}] = 0, \quad i = 1, 2,$$

and $\vec{\zeta} = (\zeta^1, \dots, \zeta^d)^T$.

Moreover, we should also require that there exist the operators \mathcal{L}_i which are linear in λ and such that

$$\mathcal{L}_i \Phi = 0, \quad i = 1, 2.$$

Then one should extract A_i and B_i for Proposition 1 from these \mathcal{L}_i rather than from the original \mathcal{X}_i .

The study of specific examples strongly suggests that the nonlocal variables ψ , χ and $\vec{\zeta}$ usually do not mix and enter Φ linearly, so it typically suffices to look for nonlocal symmetries Φ of $\mathcal{F} = 0$ in one of the following forms,

$$\Phi^\alpha = \sum_{s=1}^r a_s^\alpha \psi^s + \sum_{s=1}^r \sum_{k=1}^d a_s^k D_{x^k}(\psi^s), \quad (14)$$

$$\Phi^\alpha = \sum_{s=1}^r b^{\alpha,s} \chi_s + \sum_{s=1}^r \sum_{k=1}^d b^{\alpha,s,k} D_{x^k}(\chi_s), \quad (15)$$

$$\Phi^\alpha = \sum_{j=1}^d c_j^\alpha \zeta^j + \sum_{k,l=1}^d c_l^{\alpha,k} D_{x^k}(\zeta^l), \quad (16)$$

where $\alpha = 1, \dots, N$, $p \in \mathbb{N}$, a_s^α , $a_s^{\alpha,k}$, $b^{\alpha,s}$, $b^{\alpha,s,k}$, c_j^α and $c_l^{\alpha,k}$ are local functions, and r is the number of components of ψ , instead of general form (13). Moreover, in many cases it is possible to restrict oneself to considering only zero-order terms in the nonlocal variables, i.e., use the simpler Ansätze $\Phi^\alpha = \sum_{s=1}^r a_s^\alpha \psi^s$, $\Phi^\alpha = \sum_{s=1}^r b^{\alpha,s} \chi_s$, and $\Phi^\alpha = \sum_{j=1}^d c_j^\alpha \zeta^j$ instead of (14), (15) and (16).

A similar approach, with symmetries replaced by cosymmetries and Proposition 1 by Proposition 2 can, of course, be applied to the construction of adjoint recursion operators.

Finally, let us point out that the assumption of linear dependence of the Lax operators on λ is not as restrictive as it seems: a great many of known today multidimensional dispersionless hierarchies include systems with this property and, moreover, by Proposition 3 if a system admits an RO of the form described in Proposition 1 then it necessarily possesses a Lax-type representation whose operators are linear in λ .

Moreover, in some cases the nonlinearity in λ can be removed upon a proper rewriting of the Lax pair.

Consider for example the Pavlov equation [43]

$$u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx} = 0, \quad (17)$$

which possesses a Lax pair of the form [43] (cf. also [26])

$$\psi_y = (-u_x - \lambda)\psi_x, \quad \psi_t = (-\lambda^2 - \lambda u_x + u_y)\psi_x \quad (18)$$

which is quadratic in λ .

However, rewriting the second equation of (18) as

$$\psi_t = \lambda(-\lambda - u_x)\psi_x + u_y\psi_x$$

and substituting ψ_y for $(-\lambda - u_x)\psi_x$, we obtain

$$\psi_t = \lambda\psi_y + u_y\psi_x,$$

and thus (18) can be rewritten in the form linear in λ :

$$\psi_y = (-u_x - \lambda)\psi_x, \quad \psi_t = -\lambda\psi_y + u_y\psi_x. \quad (19)$$

4 Examples

Example 1

It is readily checked that if ψ satisfies (19) then $\Phi = 1/\psi_x$ is a nonlocal symmetry for (17), i.e., $U = 1/\psi_x$ satisfies the linearized version of (17),

$$U_{yy} + U_{xt} + u_x U_{xy} - u_y U_{xx} + u_{xy} U_x - u_{xx} U_y = 0. \quad (20)$$

Now by virtue of (19) $\Phi = 1/\psi_x$ also satisfies a pair of linear equations of the form $\mathcal{L}_i\Phi = 0$, $i = 1, 2$, where

$$\mathcal{L}_1 = -D_y + (\lambda - u_x)D_x + u_{xx}, \quad \mathcal{L}_2 = D_t + \lambda D_y - u_y D_x + u_{xy}.$$

Now, as $\Phi = 1/\psi_x$ satisfies (20) we can identify \mathcal{L}_i with \mathcal{L}_i from (11) and put $\mathcal{L}_i = \lambda A_i - B_i$, where

$$A_1 = D_x, \quad A_2 = D_y, \quad B_1 = D_y + u_x D_x - u_{xx}, \quad B_2 = -D_t + u_y D_x - u_{xy}.$$

Then it is easily seen that these operators satisfy the conditions of Proposition 1 for suitably chosen L and M , so we arrive at the recursion operator for the Pavlov equation given by the formulas

$$\tilde{U}_x = U_y + u_x U_x - u_{xx} U, \quad \tilde{U}_y = -U_t + u_y U_x - u_{xy} U,$$

which maps a (possibly nonlocal) symmetry U to a new nonlocal symmetry \tilde{U} . This is nothing but the recursion operator found in [26] rewritten as a Bäcklund auto-transformation for (20).

Example 2

Consider the general heavenly equation

$$au_{xy}u_{zt} + bu_{xz}u_{yt} + cu_{xt}u_{yz} = 0, \quad a + b + c = 0, \quad (21)$$

where a, b, c are constants. Here $d = 3$, $N = 1$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, $u^1 = u$. Note that this equation describes *inter alia* a class of anti-self-dual solutions of the Einstein field equations as shown in [22].

By definition, a (nonlocal) symmetry U of (21) satisfies the linearized version of (21), that is,

$$au_{xy}U_{zt} + au_{zt}U_{xy} + bu_{xz}U_{yt} + bu_{yt}U_{xz} + cu_{xt}U_{yz} + cu_{yz}U_{xt} = 0. \quad (22)$$

Eq.(21) admits [10, 22] a Lax-type representation with the operators

$$\begin{aligned} L_1 &= (1 + c\lambda)D_x - \frac{u_{xz}}{u_{zt}}D_t - c\lambda\frac{u_{xt}}{u_{zt}}D_z, \\ L_2 &= (1 - b\lambda)D_y - \frac{u_{yz}}{u_{zt}}D_t + b\lambda\frac{u_{yt}}{u_{zt}}D_z. \end{aligned} \quad (23)$$

It is readily seen that ψ that satisfies $L_i\psi = 0$, $i = 1, 2$, also satisfies (22), i.e., it is a nonlocal symmetry for (21), so in spirit of Proposition 3 let

$$\begin{aligned} A_1 &= cD_x - c\frac{u_{xt}}{u_{zt}}D_z, \quad A_2 = -bD_y + b\frac{u_{yt}}{u_{zt}}D_z, \\ B_1 &= -D_x + \frac{u_{xz}}{u_{zt}}D_t, \quad B_2 = -D_y + \frac{u_{yz}}{u_{zt}}D_t. \end{aligned}$$

Then all conditions of Proposition 1 are readily seen to be satisfied, e.g. we have $B_1A_2 - B_2A_1 = (1/u_{zt})\ell_F$, where $F = au_{xy}u_{zt} + bu_{xz}u_{yt} + cu_{xt}u_{yz}$.

Hence the relations

$$\tilde{U}_x = \frac{u_{xz}U_t + cu_{xt}\tilde{U}_z - u_{zt}U_x}{cu_{zt}}, \quad \tilde{U}_y = -\frac{u_{yz}U_t - bu_{yt}\tilde{U}_z - u_{zt}U_y}{bu_{zt}}, \quad (24)$$

where U is a (possibly nonlocal) symmetry of (21), define a new symmetry \tilde{U} and thus a novel recursion operator (24) for (21), i.e., a Bäcklund auto-transformation for (22). This operator has first appeared in the very first version of the present paper (arXiv:1501.01955v1) and was later rediscovered, in a slightly different form, in [23].

Example 3

Consider the Martínez Alonso–Shabat [29] equation

$$u_{yt} = u_z u_{xy} - u_y u_{xz} \quad (25)$$

where $d = 4$, $N = 1$, $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, $u^1 = u$. Eq.(21) admits [35] a Lax-type representation with the operators

$$L_1 = D_y - \lambda u_y D_x, \quad L_2 = D_z - \lambda u_z D_x + \lambda D_t. \quad (26)$$

Consider the action of operators

$$\mathcal{L}_i = \text{ad}_{L_i} = [L_i, \cdot]$$

on operators of the form $w D_x$ and spell it out as

$$\text{ad}_{L_i}(w D_x) \equiv ((\lambda B_i - A_i)w) D_x,$$

so we have

$$A_1 = D_y, \quad A_2 = D_z, \quad B_1 = u_y D_x - u_{xy}, \quad B_2 = u_z D_x - D_t - u_{xz}.$$

All conditions of Proposition 1 are again satisfied, so the relations

$$\begin{aligned} \tilde{U}_y &= u_y U_x - u_{xy} U, \\ \tilde{U}_z &= u_z U_x - u_{xz} U - U_t, \end{aligned} \quad (27)$$

where U is any (possibly nonlocal) symmetry of (25), define a new symmetry \tilde{U} , i.e., Eq.(27) provides a recursion operator for (25). This operator was already found in [35].

Applying (27) to the simplest symmetry $U = u_x$ we obtain (modulo an arbitrary function of x and t resulting from the integration) a nonlocal symmetry of the form $\tilde{U} = w - u_x^2/2$, where the nonlocal variable w is defined by the relations

$$w_y = u_y u_{xx}, \quad w_z = u_z u_{xx} - u_{xt}.$$

This example suggests that Eq.(27) can be further simplified: a new symmetry \tilde{U} can be constructed by putting

$$\tilde{U} = W - u_x U,$$

where W is defined by the relations

$$\begin{aligned} W_y &= u_y U_x + u_x U_y, \\ W_z &= u_z U_x + u_x U_z - U_t. \end{aligned}$$

Thus, the recursion operators produced within our approach are not necessarily in the simplest possible form.

Example 4

Consider a system [13, 14]

$$m_t = n_x + n m_r - m n_r, \quad n_z = m_y + m n_s - n m_s, \quad (28)$$

where x, y, z, r, s, t are independent and m, n are dependent variables.

Eq.(28) can be written as a condition of commutativity for the following pair of vector fields,

$$[D_z - m D_s - \lambda D_x + \lambda m D_r, D_y - n D_s - \lambda D_t + \lambda n D_r] = 0. \quad (29)$$

where λ is a spectral parameter, and is therefore integrable.

By virtue of (28) there exists a potential u such that

$$m = u_z/u_s, \quad n = u_y/u_s. \quad (30)$$

Substituting this into (28) gives a single second-order equation for u ,

$$u_s u_{zt} - u_z u_{st} - u_s u_{xy} + u_y u_{sx} - u_y u_{rz} + u_z u_{ry} = 0, \quad (31)$$

which can be written as a commutativity condition $[L_1, L_2] = 0$ for

$$L_1 = D_z - \frac{u_z}{u_s} D_s - \lambda D_x + \lambda \frac{u_z}{u_s} D_r, \quad L_2 = D_y - \frac{u_y}{u_s} D_s - \lambda D_t + \lambda \frac{u_y}{u_s} D_r.$$

Let χ satisfy $L_i^\dagger \chi = 0$, $i = 1, 2$. Then $\zeta = u_s/\chi$ is a nonlocal symmetry for (31) and we can readily obtain a linear system for ζ of the form $\mathcal{L}_1 \zeta = 0, \mathcal{L}_2 \zeta = 0$ from that for χ .

Upon spelling out \mathcal{L}_i as $\mathcal{L}_i = A_i - \lambda B_i$ and checking that the conditions of Theorem 1 are satisfied for suitable L and M we arrive at the following novel RO for (31):

$$\begin{aligned} \tilde{U}_y &= \frac{u_y}{u_s} \tilde{U}_s - \frac{u_y}{u_s} U_r + U_t - \frac{(u_{st} - u_{ry})}{u_s} U, \\ \tilde{U}_z &= \frac{u_z}{u_s} \tilde{U}_s - \frac{u_z}{u_s} U_r + U_x + \frac{(u_{rz} - u_{sx})}{u_s} U. \end{aligned} \quad (32)$$

Here again U is a (in general nonlocal) symmetry of (31), and \tilde{U} is a new nonlocal symmetry for (31).

Example 5

Consider (see e.g. [1, 2, 31] and references therein) the following Lax operators:

$$L_1 = D_y + \partial_{\tilde{x}} K - \lambda D_{\tilde{x}}, \quad L_2 = D_x + \partial_{\tilde{y}} K - \lambda D_{\tilde{y}}. \quad (33)$$

The commutativity condition $[L_1, L_2] = 0$ yields the system

$$\partial_x \partial_{\tilde{x}} K - \partial_y \partial_{\tilde{y}} K - [\partial_{\tilde{x}} K, \partial_{\tilde{y}} K] = 0. \quad (34)$$

Here K takes values in a (matrix) Lie algebra \mathfrak{g} and is known as the Yang K -matrix.

Note that (34) can be rewritten in the dispersionless form of the type (1) upon introducing (cf. e.g. [31, 42]) an additional dependent variable J taking values in $G = \exp(\mathfrak{g})$. Namely, the dispersionless form of (34) reads

$$\partial_x J = J \partial_{\tilde{y}} K, \quad \partial_y J = J \partial_{\tilde{x}} K. \quad (35)$$

The compatibility condition $\partial_x \partial_y J = \partial_y \partial_x J$ yields an equation

$$\partial_{\tilde{x}}(J^{-1} \partial_x J) = \partial_{\tilde{y}}(J^{-1} \partial_y J), \quad (36)$$

while (34) is identically satisfied by virtue of (35). In other words, (35) defines a Bäcklund transformation relating (34) and (36).

If we define the gauge potentials by setting $A_x = \partial_{\tilde{y}} K$, $A_y = \partial_{\tilde{x}} K$, $A_{\tilde{x}} = 0$, $A_{\tilde{y}} = 0$, then they satisfy the anti-self-dual Yang–Mills equations on \mathbb{R}^4 with Euclidean metric with coordinates X^i such that $\sqrt{2}x = X^1 + iX^2$, $\sqrt{2}\tilde{x} = X^1 - iX^2$, $\sqrt{2}y = X^3 - iX^4$, $\sqrt{2}\tilde{y} = X^3 + iX^4$, where $i = \sqrt{-1}$, cf. e.g. [31]. Conversely, any sufficiently smooth solution of the anti-self-dual Yang–Mills equations can, up to

a suitable gauge transformation, be obtained in this way [31]. In other words, the anti-self-dual Yang–Mills equations, which play an important role in modern physics, cf. e.g. [31] and references therein, can, in a suitable gauge, be written in dispersionless form, exactly as claimed in Introduction.

It is natural to assume that in the Lax pair equations $L_i\psi = 0$ we have $\psi \in G = \exp(\mathfrak{g})$. However, such ψ cannot be a symmetry for (34), since a symmetry must live in \mathfrak{g} just like K . However, we can get a Lax representation for $\Phi \in \mathfrak{g}$ using the adjoint action: $[L_1, \Phi] = [L_2, \Phi] = 0$, i.e., we use the Lax operators $\mathcal{L}_i = \text{ad } L_i$.

Then all conditions of Proposition 1 are satisfied for A_i and B_i obtained by spelling out \mathcal{L}_i as $\mathcal{L}_i = \lambda A_i - B_i$, which yields the following RO for (34):

$$V_{\tilde{x}} = U_y + [\partial_{\tilde{x}}K, U], \quad V_{\tilde{y}} = U_x + [\partial_{\tilde{y}}K, U], \quad (37)$$

where U and V are symmetries for (34). This RO is known from the literature, see [6, 42, 2].

If in the above construction we take $\mathfrak{g} = \mathfrak{diff}(\mathbb{R}^N)$, i.e., we put

$$K = \sum_{i=1}^N u_i \partial / \partial z^i,$$

where $u_i = u_i(x, y, \tilde{x}, \tilde{y}, z^1, \dots, z^N)$ are scalar functions, then (34) becomes [28] the Manakov–Santini system [25], an integrable system in $(N + 4)$ independent variables, and (37) provides a recursion operator for this system which was found earlier in [30] (see also [28]) by a different method.

Further examples of recursion operators obtained using our approach can be found in [19].

5 Concluding remarks

We presented above a method of construction for recursion operators and adjoint recursion operators for a broad class of multidimensional integrable systems that can be written as commutativity conditions for a pair of vector fields, or, even more broadly, of linear combinations of vector fields with zero-order matrix operators, linear in the spectral parameter and free of derivatives in the latter. This method is based on constructing a special Lax-type representation for the system under study from the original Lax-type representation, see Section 3 for details. Note that if a system under study admits several essentially distinct Lax representations then our approach applied to these Lax representations in principle can give rise to several essentially distinct recursion operators.

To the best of author’s knowledge, the method in question works for all known examples of multidimensional dispersionless integrable systems admitting recursion operators in the form of Bäcklund auto-transformation for linearized systems, including e.g. the ABC equation [48]

$$au_xu_{yt} + bu_yu_{xt} + cu_tu_{xy} = 0, \quad a + b + c = 0,$$

the simplest (2+1)-dimensional equation of the so-called universal hierarchy of Martínez Alonso and Shabat [29],

$$u_{yy} - u_yu_{tx} + u_xu_{ty} = 0,$$

the complex Monge–Ampère equation (see [38] for its recursion operator), first, second and modified heavenly equations, etc. In particular, this method enabled us to find hitherto unknown recursion operators for the general heavenly equation of Doubrov and Ferapontov (21) and for the equation (31). Note that our approach, when applicable, is in general computationally less demanding than the methods of [30] and [34].

While all examples to which we have applied our approach so far are dispersionless, Propositions 1 and 2 and the method described in Section 3 do not explicitly require this to be the case, so it would be interesting to find out whether it is possible to extend our approach to the construction of recursion operators to other classes of integrable systems.

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